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A RUBBER WEDGE UNDER THE COMPRESSION OF A LINE LOAD

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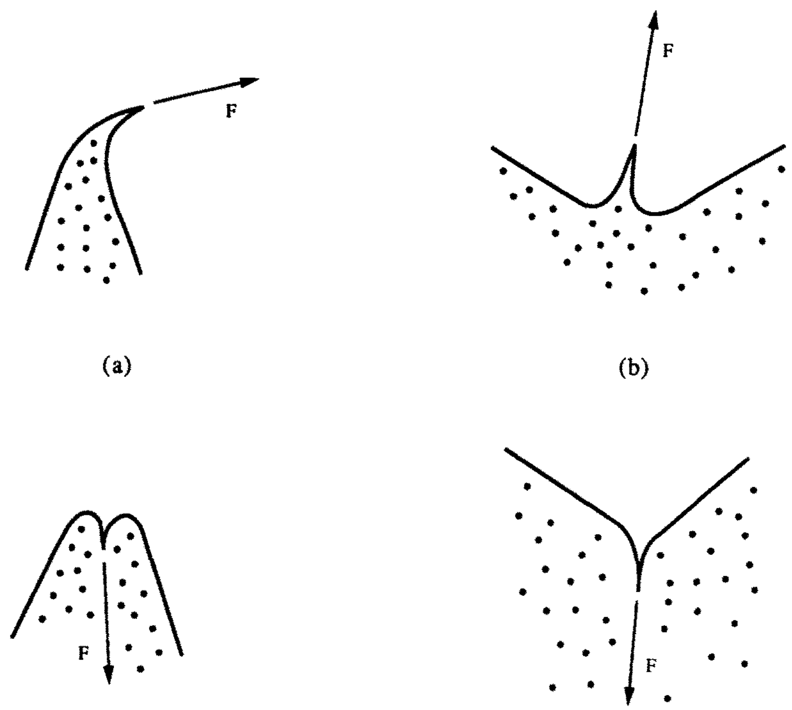
Abstract—Using the constitutive equation of rubber-like materials given by Gao (1990, *Theoretical Appl. Fract. Mech.* **14,** 219–231), this paper investigates the problem of a wedge (or notch) under the compression of a line load at its tip. The problem was treated as a plane strain case and the large deformation was taken into account. The asymptotic solution to the stress–strain field near a wedge apex (or a notch corner) is obtained. It is revealed that under compression of a line load, a notch corner will be locally closed, i.e. the notch surface will contact each other. Under a similar load, the wedge tip will be shrivelled to form a locally closed notch, i.e. the V-shaped wedge apex will become W-shaped.

1. INTRODUCTION

In linear elastic theory, the problem of a concentrated line force acting on a wedge tip (or a notch corner) is a very typical one and its solution can be found in very early literature (Michell, 1902). According to the linear solution (Michell, 1902), when the wedge tip (or notch corner) is approached, the stress and strain will tend to infinity, i.e. the field possesses singularity. Although the singular field contains a large deformation area, for ordinary engineering materials people can only consider the small deformation domain. Rubber-like materials can sustain large strain, therefore the stress–strain behavior of singular field must be analysed by nonlinear theory. There are two obstacles for solving the nonlinear singular field: firstly, the inherently nonlinear geometry is difficult to describe; secondly, when the singular point is approached, because of the tremendously large strain, the ordinary constitutive relation may become physically invalid. Gao (1990) gave a constitutive relation for rubber-like materials in which the stress tensor was decomposed into a spherical part and a deviatoric part. This constitutive relation kept reasonably when strain tended to infinity, and it was used to analyse the singular field of a plane strain crack. Knowles and Sternberg (1973, 1974) gave an elastic constitutive relation which contained three parameters. Their constitutive relation was also used to analyse the singular field near a plane strain crack tip, but the classification of the solution is complicated and the second approximation is needed. Furthermore, Gao's constitutive relation was verified to be also reasonable and convenient for analysing the plane stress crack tip field (Gao and Durban, 1994). Using Gao's constitutive relation, Gao (1994) recently gave an asymptotic solution to the singular field near a wedge tip (or a notch corner) of rubber-like materials that is under the tension of a line load, i.e. the force directs outside the material domain as shown in Figs 1(a, b). However, for a similar problem, if the force directs inside the material domain, i.e. the wedge is under compression as shown in Figs 1(c, d), the solution cannot be obtained by simply changing the signs of every quantity. Actually, the compression and tension are quite different problems in large deformation cases; the deformation patterns and field singularities are different for both cases. Using the constitutive relation given in Gao (1990), the present paper analyses the singular field near a wedge tip (or a notch corner) under the compression of a line load as shown in Figs 1(c, d). By dividing the field into expanding sectors and narrowing sectors, the asymptotic solution to the wedge tip (or notch corner) is obtained. The result shows that under compression, a notch will be locally closed as shown in Fig. 1(d), while a wedge tip will be shrivelled to form a locally closed



(a) (b)

(c) (d)

Fig. 1. Various loading cases: (a) wedge under tension; (b) a notch under tension; (c) a wedge under compression; (d) a notch under compression.

notch as shown in Fig. 1(c). For simplicity, we only analyse the problem of a wedge under compression, but the same analysis is valid for a notch.

2. BASIC EQUATIONS

Considering a three dimensional domain of material, we denote the position vectors of a material point by **P** and **p**, before and after loading, respectively. x^i $(i = 1, 2, 3)$ denotes the Lagrangian coordinate, then we can define two local triads;

$$\mathbf{P}_i = \frac{\partial \mathbf{P}}{\partial x^i}, \quad \mathbf{p}_i = \frac{\partial \mathbf{p}}{\partial x^i}. \tag{1}$$

The displacement gradient is

$$\mathbf{F} = \mathbf{p}_i \otimes \mathbf{P}^i \tag{2}$$

in which the summation rule is implied, $\mathbf{P}^i$ is the conjugate base of $\mathbf{P}_i$, $\otimes$ is the dyadic symbol. The right and left Cauchy–Green strain **D** and **d** are

$$\mathbf{D} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}, \quad \mathbf{d} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} \tag{3}$$

where superscript T indicates transposition. It can be proved that **D** and **d** possess the same invariants

$$I_1 = \mathbf{D} : \mathbf{E} = \mathbf{d} : \mathbf{E}$$

$$I_2 = \mathbf{D}^2 : \mathbf{E} = \mathbf{d}^2 : \mathbf{E}$$

$$I_3 = \mathbf{D}^3 : \mathbf{E} = \mathbf{d}^3 : \mathbf{E} \tag{4}$$

in which ":" denotes dual scale product and **E** is unit tensor.

In Gao (1990), a new form of specific strain energy is given,

$$W = A[(I/K^{1/3})^n - 3^n] + B(K-1)^m K^{-q} \tag{5}$$

where

$$I = I_1, \quad K = \tfrac{1}{6}(I_1^3 - 3I_1 I_2 + 2I_3), \tag{6}$$

A, B, n, m, q are positive constants. The Kirchhoff stress σ and Cauchy stress τ can be obtained from W

$$\boldsymbol{\sigma} = 2\frac{\partial W}{\partial \mathbf{D}}, \quad \boldsymbol{\tau} = K^{-1/2}\mathbf{F}\cdot\boldsymbol{\sigma}\cdot\mathbf{F}^{\mathrm{T}}. \tag{7}$$

From eqns (5)–(7) we obtain

$$\boldsymbol{\tau} = 2nAI^{n-1}K^{-1/2-(n/3)}\left(\mathbf{d} - \frac{I}{3}\mathbf{E}\right) + 2B(K-1)^{m-1}K^{-q-1/2}[(m-q)K+q]\mathbf{E}. \tag{8}$$

Evidently, the stress in eqn (8) is decomposed into the deviatoric part and spherical part. This is the main advantage of relation (8). It is verified that if $m > q$, $3q > n$ and $n > 3/4$, the relation (8) keeps reasonable for a singular point. The equilibrium equation is

$$\nabla\cdot\boldsymbol{\tau} = 0 \tag{9}$$

in which

$$\nabla = \mathbf{p}^i \frac{\partial}{\partial x^i}. \tag{10}$$

3. EXPANDING SECTOR

We can imagine various deformation patterns of the wedge tip under compression, from which various mathematical formula were obtained. However, only one is reasonable, as stated in the following. Figure 2(a) shows the cross section of a wedge before loading, whereas Fig. 2(b) shows the same cross section after loading. For simplicity, only the symmetric loading case is considered here. The wedge tip field can be divided into several sectors. N and N' are called narrowing sectors; before deformation, N and N' occupy almost the whole wedge tip domain, but after deformation they become very narrow. E is called the expanding sector; before deformation E is very narrow, but after deformation it becomes very wide and occupies almost the whole domain surrounding the wedge tip. The

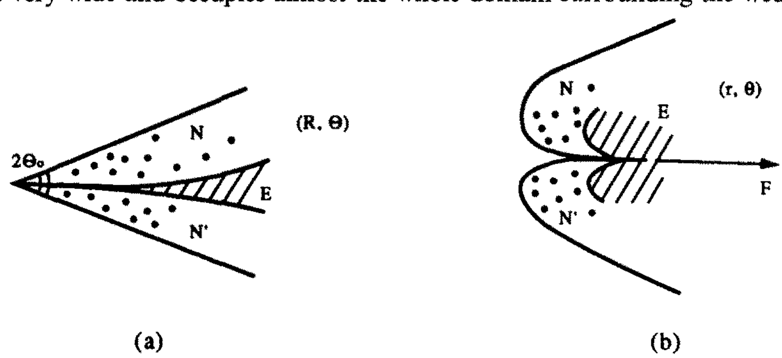


Fig. 2. Deformation pattern of a wedge: (a) before loading; (b) after loading.

deformation pattern in domain N (or N') and E are quite different so that they must be described individually.

3.1. *Strain and stress*

Now we consider the expanding sector E. Two Lagrangian coordinate systems, (R, Θ) and (r, θ), are taken so that (R, Θ) is the polar coordinate before deformation, while (r, θ) is the polar coordinate after deformation. In sector E we assume that (R, Θ) are given by the following mapping function of (r, θ)

$$\begin{aligned} R &= r^{1-d}\varphi(\theta) \\ \Theta &= r^{c}\psi(\theta) \end{aligned} \tag{11}$$

where d, c are constants to be determined. Let

$$\begin{aligned} \mathbf{P}_r &= \frac{\partial \mathbf{P}}{\partial r}, \quad \mathbf{P}_\theta = \frac{\partial \mathbf{P}}{\partial \theta} \\ \mathbf{e}_R &= \frac{\partial \mathbf{P}}{\partial R}, \quad \mathbf{e}_\Theta = \frac{1}{R}\frac{\partial \mathbf{P}}{\partial \Theta}, \end{aligned} \tag{12}$$

then, from eqns (11) and (12), we can obtain

$$\begin{aligned} \mathbf{P}_r &= r^{-d}\varphi[(1-d)\mathbf{e}_R + cr^{c}\psi\mathbf{e}_\Theta] \\ \mathbf{P}_\theta &= r^{1-d}(\varphi'\mathbf{e}_R + r^{c}\varphi\psi'\mathbf{e}_\Theta). \end{aligned} \tag{13}$$

Therefore the conjugate base is obtained

$$\begin{aligned} \mathbf{P}^r &= r^{d-c}(r^{c}\varphi\psi'\mathbf{e}_R - \varphi'\mathbf{e}_\Theta)/S \\ \mathbf{P}^\theta &= r^{d-c-1}\varphi[-cr^{c}\psi\mathbf{e}_R + (1-d)\mathbf{e}_\Theta]/S, \end{aligned} \tag{14}$$

in which

$$S = \varphi[(1-d)\varphi\psi' - c\varphi'\psi]. \tag{15}$$

Let

$$\mathbf{e}_r = \frac{\partial \mathbf{p}}{\partial r} \quad \text{and} \quad \mathbf{e}_\theta = \frac{1}{r}\frac{\partial \mathbf{p}}{\partial \theta}, \tag{16}$$

then from eqns (2), (3), (14) and (16) we obtain the dominant terms of $\mathbf{d}$;

$$\mathbf{d} = r^{2d-2c}S^{-2}[\varphi'^2\mathbf{e}_r\otimes\mathbf{e}_r + (1-d)^2\varphi^2\mathbf{e}_\theta\otimes\mathbf{e}_\theta - (1-d)\varphi\varphi'(\mathbf{e}_r\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_r)]. \tag{17}$$

Using eqns (4), (6) and (17) it follows

$$K^{1/2} = r^{2d-c}S^{-1}, \quad I = r^{2d-2c}TS^{-2} \tag{18}$$

in which

$$T = \varphi'^2 + (1-d)^2\varphi^2. \tag{19}$$

Since the wedge tip is under compression, we can assume that $K \to 0$ when $r \to 0$. This assumption is verified by the analysis process, then eqn (8) is simplified to

$$\tau = 2nAI^{n-1}K^{-(n/3)-1/2}\left(\mathbf{d}-\frac{I}{3}\mathbf{E}\right)-2qBK^{-q-1/2}\mathbf{E}. \tag{20}$$

Substituting eqns (17) and (18) into eqn (20) and matching the singularity we obtain

$$\tau = r^{-\mu}\left\{2nAT^{n-1}S^{1-(4n/3)}\left[\left(\varphi'^2-\frac{T}{3}\right)\mathbf{e}_r\otimes\mathbf{e}_r+\left((1-d)^2\varphi^2-\frac{T}{3}\right)\mathbf{e}_\theta\otimes\mathbf{e}_\theta\right.\right.$$
$$\left.\left.-(1-d)\varphi\varphi'(\mathbf{e}_r\otimes\mathbf{e}_\theta+\mathbf{e}_\theta\otimes\mathbf{e}_r)\right]-2qBS^{2q+1}\mathbf{E}\right\} \tag{21}$$

and

$$\mu = \frac{3n(2q+1)}{3q+2n}d, \quad c = \frac{6q+n}{3q+2n}d. \tag{22}$$

In order to match the order of stress with that of load, it is required that $\mu = 1$, then we have

$$d = \frac{2n+3q}{3n(2q+1)}, \quad c = \frac{n+6q}{3n(2q+1)}. \tag{23}$$

3.2. *Equilibrium equation and solution*

The equilibrium eqn (9) is written as

$$\frac{\partial\tau^{rr}}{\partial r}+\frac{\partial\tau^{r\theta}}{r\,\partial\theta}+\frac{1}{r}(\tau^{rr}-\tau^{\theta\theta}) = 0$$
$$\frac{\partial\tau^{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\tau^{\theta\theta}}{\partial\theta}+\frac{2}{r}\tau^{r\theta} = 0 \tag{24}$$

where

$$\tau^{rr} = r^{-1}\left[2nAT^{n-1}S^{1-(4n/3)}\left(\varphi'^2-\frac{T}{3}\right)-2qBS^{2q+1}\right]$$
$$\tau^{\theta\theta} = r^{-1}\left[2nAT^{n-1}S^{1-(4n/3)}\left((1-d)^2\varphi^2-\frac{T}{3}\right)-2qBS^{2q+1}\right]$$
$$\tau^{r\theta} = -r^{-1}[2nAT^{n-1}S^{1-(4n/3)}(1-d)\varphi\varphi']. \tag{25}$$

Since $\tau^{rr}\sim\tau^{r\theta}\sim\tau^{\theta\theta}\sim r^{-1}$, the solution of eqn (24) takes the form,

$$\tau^{\theta\theta} = r^{-1}At\cos\theta, \quad \tau^{r\theta} = r^{-1}At\sin\theta, \tag{26}$$

where t is a constant to be determined by load. From eqns (25) and (26) we obtain the equations to determine φ and ψ;

$$\frac{-t\sin\theta}{3(1-d)\varphi\varphi'}[2(1-d)^2\varphi^2-\varphi'^2]-2q\frac{B}{A}\cdot S^{2q+1} = t\cos\theta,$$
$$2n(1-d)\varphi\varphi' T^{n-1}S^{1-(4n/3)} = -t\sin\theta. \tag{27}$$

In addition we obtain the expression of τ^{rr}

$$\tau^{r\prime} = r^{-1}t\left\{\cos\theta + \frac{\sin\theta}{(1-d)\varphi\varphi'}[(1-d)^2\varphi^2 - \varphi'^2]\right\}. \tag{28}$$

The boundary conditions for eqn (27) are

$$\varphi'(0) = 0 \tag{29}$$

$$\psi(0) = 0. \tag{30}$$

For a given value of t, eqn (27) under boundary conditions (29) and (30), has a unique solution, φ and ψ. As mentioned above, t is a parameter depending upon the load, therefore it is necessary to discuss the general relation between φ, ψ and t. From eqns (27), (29) and (30), it is easy to prove that if φ, ψ are the solutions corresponding to the parameter t, then $k^d\varphi$, $k^{-c}\psi$ will be the solutions corresponding to the parameter kt. Here k is an arbitrary positive constant. So, when parameter t changes, the solutions φ and ψ will undergo a similar transformation; t can be considered as the controlling parameter of the field. Based on the above discussion, we can calculate eqn (27) only for a specific value of t (for example, $t = 1$). Since eqn (27) is nonlinear for the derivative φ', for convenience of numerical solution, from eqn (27) we can obtain

$$\varphi'' = \frac{\varphi'^2}{\varphi} - \frac{T\varphi'}{\Omega}\left\{L\left[2nM\frac{\varphi'}{\varphi} - (1+M)\cot\theta\right] + \frac{3(1-d)}{\sin^2\theta}\varphi\varphi'\right\} \tag{31}$$

where

$$M = \frac{3(2q+1)}{4n-3}, \quad L = 3(1-d)\varphi\varphi'\cot\theta + 2(1-d)^2\varphi^2 - \varphi'^2$$

$$\Omega = T[T+(1-d)^2\varphi^2] + ML[(2n-1)\varphi'^2 + (1-d)^2\varphi^2]. \tag{32}$$

For eqn (31), we must supplement a new boundary condition

$$\varphi(0) = \varphi_0 \tag{33}$$

where φ_0 is a constant to adjust the given value of t. If we let $\varphi = \varphi_0 \cdot \Phi(\theta)$, then $\Phi(0) = 1$; φ_0 expresses the magnitude of deformation field. The relation between φ_0 and t will be discussed later. The curves of $\Phi(\theta)$ for some values of n and q are shown in Fig. 3. The shape of curves $\Phi(\theta)$ is independent of parameter t.

From the second of eqn (27), the equation of ψ is obtained

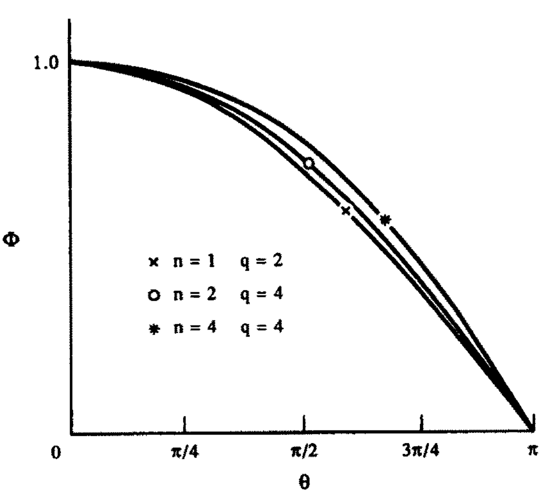


Fig. 3. The curves of $\Phi = \varphi/\varphi_0$ for $B/A = 1$.

Table 1. $C\varphi/\varphi_0$

	q		
n	2	4	8
1	0.5004	0.4995	0.4992
2	0.5288	0.5443	0.5552
4	0.5807	0.6191	0.6475
10	0.6303	0.6949	0.7453

$$(1-d)\varphi\psi' - c\varphi'\psi = \frac{1}{\varphi}\left[-\frac{2n(1-d)\varphi\varphi'}{t\cdot\sin\theta}T^{n-1}\right]^{3/(4n-3)}. \tag{34}$$

When φ_0 is fixed, eqn (34), under condition (30), can be calculated numerically with eqn (31) if t is known. Unfortunately, for a fixed φ_0 we still do not know the corresponding t, and eqn (27) does not provide any relation between φ_0 and t since $\varphi'(0) = 0$. In order to reveal the relation of φ_0 and t we go back to the original eqn (27) and consider its asymptotic behavior when $\theta \to \pi$. It is easy to prove that $\varphi \sim (\pi-\theta)$, $\psi \sim (\pi-\theta)^{-1}$ when $\theta \to \pi$. Therefore we can let

$$\varphi = C_\varphi(\pi-\theta), \quad \psi = C_\psi(\pi-\theta)^{-1} \quad \text{when} \quad \theta \to \pi. \tag{35}$$

By substituting eqn (35) into eqn (27), we can obtain

$$\begin{aligned} C_\varphi &= 2^{-(1/2n)}\left[\frac{A}{B}\left(d-\frac{1}{2n}\right)\right]^{d-(1/2n)}\left[\frac{t}{n(1-d)}\right]^{d} \\ C_\psi &= \frac{1}{1+c-d}2^{1/n}\left[\frac{A}{B}\left(d-\frac{1}{2n}\right)\right]^{(1/n)-c}\left[\frac{t}{n(1-d)}\right]^{-c}. \end{aligned} \tag{36}$$

On the other hand, from the numerical calculation of eqn (31), we can obtain the ratio of $C_\varphi/\varphi_0 = H$ for various n and q. The value of H is listed in Table 1. Using Table 1 and eqn (36) we can calculate the value t for a given φ_0, therefore the numerical solution of eqn (34) can be performed. When $\varphi_0 = 1$ and $B/A = 1$, the curves of ψ for various n, q are shown in Fig. 4.

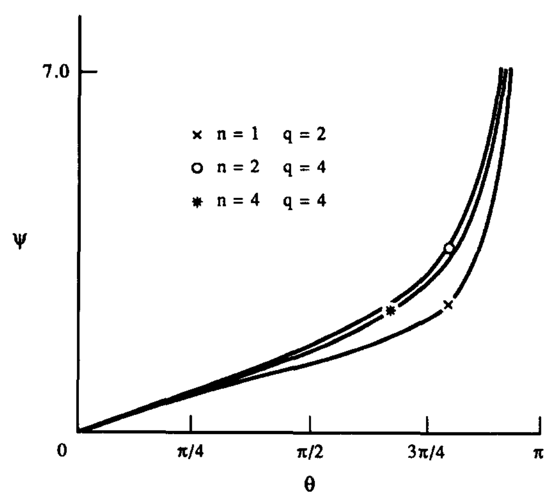


Fig. 4. The curves of ψ for $\varphi(0) = 1$, $B/A = 1$.

4. NARROWING SECTORS

The formula (11) is not valid for the vicinity of $\theta = \pm\pi$ in Fig. 2(b) because $\varphi \to 0$ and $\psi \to \infty$. Now, we consider the narrowing sectors N and N' that occupy almost the whole wedge tip domain before loading but shrink to the vicinity of $\theta = \pm\pi$ after loading. For simplicity we only discuss sector N, but the method is also valid for sector N'.

4.1. *Strain and stress*

In sector N we assume

$$R = r^{1+b} f(\xi)$$
$$\Theta = g(\xi), \quad \xi = (\pi-\theta)r^{-a}. \tag{37}$$

From eqns (12) and (37) we obtain

$$\mathbf{P}_r = r^b\{[(1+b)f - a\xi f']\mathbf{e}_R - a\xi f g' \mathbf{e}_\Theta\}$$
$$\mathbf{P}_\theta = r^{1+b-a}(-f'\mathbf{e}_R - f g'\mathbf{e}_\Theta). \tag{38}$$

Then we can obtain the conjugate base

$$\mathbf{P}^r = r^{-b}\frac{1}{v}(-fg'\mathbf{e}_R + f'\mathbf{e}_\Theta)$$
$$\mathbf{P}^\theta = r^{a-b-1}\frac{1}{v}\{a\xi f g'\mathbf{e}_R + [(1+b)f - a\xi f']\mathbf{e}_\Theta\} \tag{39}$$

in which

$$v = -(1+b)f^2 g'. \tag{40}$$

Let

$$\mathbf{d} = d^{ij}\mathbf{e}_i \otimes \mathbf{e}_j \quad (i, j = r, \theta) \tag{41}$$

then from eqns (39) and (3) we can obtain

$$d^{rr} = r^{-2b}v^{-2}u$$
$$d^{\theta\theta} = r^{2a-2b}v^{-2}\{a^2\xi^2 f^2 g'^2 + [(1+b)f - a\xi f']^2\}$$
$$d^{r\theta} = r^{a-2b}v^{-2}\{-a\xi f^2 g'^2 + f'[(1+b)f - a\xi f']\} \tag{42}$$

where

$$u = f'^2 + f^2 g'^2. \tag{43}$$

Furthermore, we obtain

$$I = r^{-2b}v^{-2}u, \quad K^{1/2} = r^{a-2b}v^{-1}. \tag{44}$$

Substituting eqns (42)–(44) into eqn (20) and matching the singularity, noting that $\tau \sim r^{-1}$, it follows

$$a = \frac{6q+n}{3n(2q+1)} = c, \quad b = \frac{3q-n}{3n(2q+1)} = c-d. \tag{45}$$

4.2. *Coordinate transformation*

Using a similar method to Gao (1994), we introduce a new coordinate system (ξ, η) in the vicinity of $\theta = \pi$, as shown in Fig. 5,

$$\begin{aligned} \xi &= (\pi-\theta)/r^a \\ \eta &= r\left[1+\frac{a}{2}(\pi-\theta)^2\right]. \end{aligned} \tag{46}$$

It can be proved that (ξ, η) is an approximately orthogonal system in the vicinity of $\theta = \pi$. The inverse expression is

$$\begin{aligned} \theta &= \pi-\xi\eta^a \\ r &= \eta\left[1-\frac{a}{2}(\xi\eta^a)^2\right]. \end{aligned} \tag{47}$$

From eqns (46) and (47) we can obtain the relations between unit vectors

$$\mathbf{e}_r = \mathbf{e}_\eta - a\xi\eta^a\mathbf{e}_\xi, \quad \mathbf{e}_\theta = -\mathbf{e}_\xi - a\xi\eta^a\mathbf{e}_\eta. \tag{48}$$

Using eqns (47), (48), (41) and (42) we obtain

$$\mathbf{d} = \eta^{-2b}v^{-2}[u\mathbf{e}_\eta \otimes \mathbf{e}_\eta - \eta^a(1+b)ff'(\mathbf{e}_\xi \otimes \mathbf{e}_\eta + \mathbf{e}_\eta \otimes \mathbf{e}_\xi) + \eta^{2a}(1+b)^2 f^2 \mathbf{e}_\xi \otimes \mathbf{e}_\xi]. \tag{49}$$

Substituting eqns (44) and (49) into eqn (20), it follows

$$\begin{aligned} \tau^{\eta\eta} &= \eta^{-1}(\tfrac{4}{3}nAu^n v^{1-(4n/3)} - 2qBv^{2q+1}) \\ \tau^{\xi\xi} &= -\eta^{-1}(\tfrac{2}{3}nAu^n v^{1-(4n/3)} + 2qBv^{2q+1}) \\ \tau^{\xi\eta} &= -\eta^{a-1}2nA(1+b)u^{n-1}v^{1-(4n/3)}ff'. \end{aligned} \tag{50}$$

Furthermore, using eqn (48) we have

$$\begin{aligned} \mathbf{e}_\xi \cdot \frac{\partial}{\partial\xi}\mathbf{e}_\eta &= -\mathbf{e}_\eta \cdot \frac{\partial}{\partial\xi}\mathbf{e}_\xi = (1+a)\eta^a \\ \mathbf{e}_\xi \cdot \frac{\partial}{\partial\eta}\mathbf{e}_\eta &= -\mathbf{e}_\eta \cdot \frac{\partial}{\partial\eta}\mathbf{e}_\xi = a(1+a)\eta^{a-1}\xi. \end{aligned} \tag{51}$$

4.3. *Equilibrium equation and solution*

Substituting eqns (51) and (50) into eqn (9) and matching the singularity we can obtain

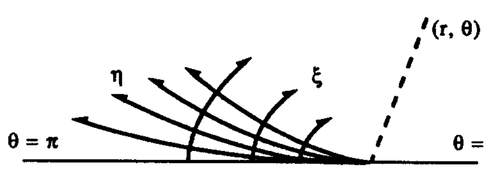


Fig. 5. ξ, η coordinate.

$$\frac{\partial}{\partial\eta}\tau^{\eta\eta}+\frac{1}{\eta^{1+a}}\frac{\partial}{\partial\xi}\tau^{\xi\eta}+\frac{1+a}{\eta}(\tau^{\eta\eta}-\tau^{\xi\xi})=0$$

$$\frac{\partial}{\partial\xi}\tau^{\xi\xi}=0. \tag{52}$$

Noting that $\tau^{\eta\eta}\sim\tau^{\xi\xi}\sim\eta^{-1}$, $\tau^{\xi\eta}\sim\eta^{a-1}$ and considering eqn (26), we can reduce eqn (52) as

$$\tau^{\xi\xi}=-\eta^{-1}t$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}[u^{n-1}v^{1-(4n/3)}(1+b)ff']-au^{n}v^{1-(4n/3)}-\frac{t}{2n}=0 \tag{53}$$

in which t is the same constant as in eqn (26). For the convenience of numerical calculation from eqn (53) we obtain

$$f''-fg'^2=\frac{u}{\Delta f}\left(1-\frac{a+tQ}{1+b}\right)\left[2nf'^2-\left(\frac{2n}{3}+1\right)u+(2q+1)(1-3tQ)u\right]$$

$$g''+2\frac{f'g'}{f}=\frac{ug'}{\Delta}\left(1-\frac{a+tQ}{1+b}\right)\frac{2nf'}{f} \tag{54}$$

in which

$$Q=\frac{1}{2n}u^{-n}v^{(4n/3)-1}$$

$$\Delta=2\left(\frac{n}{3}-1\right)f'^2+\left(\frac{2n}{3}+1\right)u-(2q+1)(1-3tQ)[2(n-1)f'^2+u]. \tag{55}$$

The boundary conditions of eqn (54) are

$$\tau^{\xi\eta}(0)=0,\quad \tau^{\xi\xi}(0)=-\eta^{-1}t \tag{56}$$

$$g(0)=\Theta_0,\quad g(\infty)=0. \tag{57}$$

Using eqn (50) we can write eqn (56) as

$$f'(0)=0,\quad \left(\frac{2n}{3}u^{n}v^{1-(4n/3)}+2q\frac{B}{A}v^{2q+1}\right)_{\xi=0}=t. \tag{58}$$

Equation (54), under boundary conditions (57) and (58), can be solved numerically; $f(0)$ and $g'(0)$ are parameters to adjust conditions (57) and (58). The curves of $f(\xi)$ and $g(\xi)$ are shown in Figs 6 and 7 for $\Theta_0=45°$, $B/A=1$, $f(0)=1$ (corresponding t is calculated),

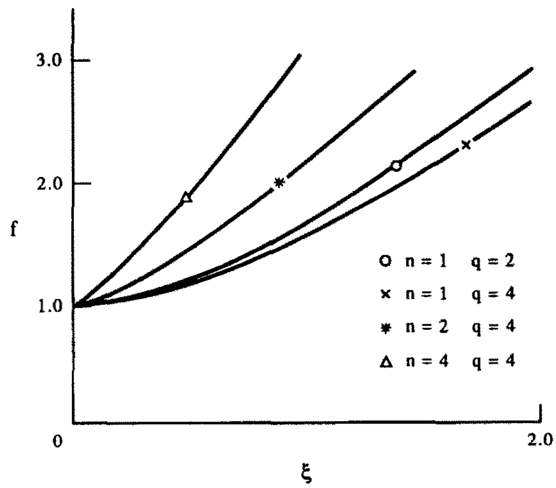


Fig. 6. The curves of $f(\xi)$ for $B/A=1$, $f(0)=1$, $\Theta_0=45°$.

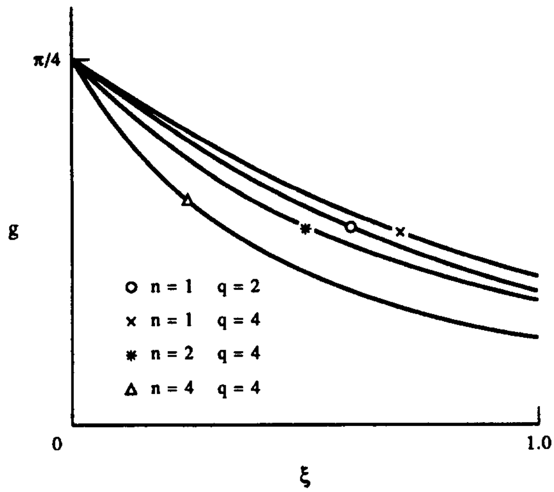


Fig. 7. The curves of $g(\xi)$ for $B/A = 1, f(0) = 1, \Theta_0 = 45°$.

but n and q are taken as various values. The curves of $f(\xi)$ and $g(\xi)$ are shown in Figs 8 and 9 for $B/A = 1, f(0) = 1, \Theta_0 = 90°$. This is the case of a half space acted by a line load.

4.4. *The nature of the solution*

Now we discuss the general nature of the solution of eqn (54). From eqns (54), (57) and (58) we can see that if f and g are the solutions corresponding to the parameter t, then f, ξ and t can be replaced by $k^{d-c}f$, $k^{-c}\ \xi$ and kt, respectively and the equations are still satisfied, i.e. when parameter t changes, the solution f and g will undergo a similar transformation. Therefore, we only need to give the numerical result of eqn (54) for a specified value of t (for example $t = 1$).

The numerical result shows that $tQ \to 1+b-a$ when $\xi \to \infty$, i.e. eqn (54) has the following asymptotic form:

$$f''-f'g'^2 = 0, \quad g''+2\frac{f'}{f}g' = 0 \quad \text{when} \quad \xi \to \infty. \tag{59}$$

The solution of eqn (59) is

$$f = f^*[\omega^2(\xi-\xi_0)^2+1]^{1/2}, \quad g = g^*-\text{arc } tg[\omega(\xi-\xi_0)] \tag{60}$$

where f^*, g^*, ω and ξ_0 are constants.

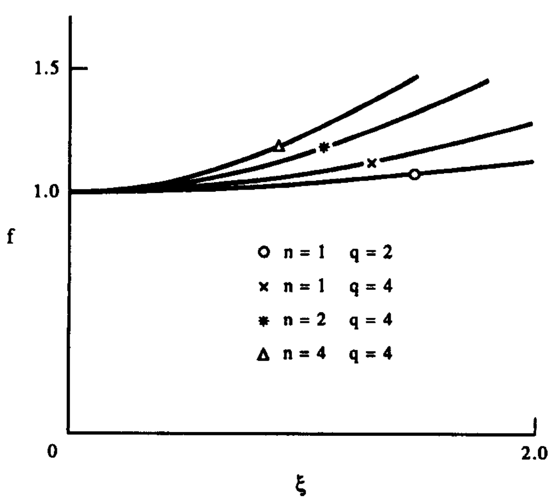


Fig. 8. The curves of $f(\xi)$ for $B/A = 1, f(0) = 1, \Theta_0 = 90°$.

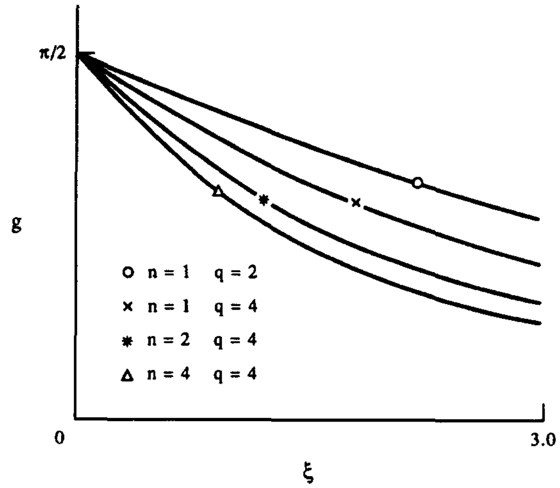


Fig. 9. The curves of $g(\xi)$ for $B/A = 1, f(0) = 1, \Theta_0 = 90°$.

Considering the boundary condition (57) from eqn (60) we have the following asymptotic expression for f and g when $\xi \to \infty$,

$$f = f^*\omega\xi, \quad g = \frac{1}{\omega\xi}. \tag{61}$$

Using eqns (40), (43) and (61) we have

$$v = (1+b)f^{*2}\omega, \quad u = f^{*2}\omega^2 \quad \text{when} \quad \xi \to \infty. \tag{62}$$

Substituting eqn (62) into the second equation of eqn (50), then into the first equation of eqn (53), we can obtain

$$\frac{2n}{3} f^{*2n}\omega^{2n}[(1+b)f^{*2}\omega]^{1-(4n/3)} + 2q\frac{B}{A}[(1+b)f^{*2}\omega]^{2q+1} = t. \tag{63}$$

On the other hand, as mentioned before, $tQ \to 1+b-a$ when $\xi \to \infty$, i.e.

$$\frac{t}{2n} f^{*-2n}\omega^{-2n}[(1+b)f^{*2}\omega]^{(4n/3)-1} = 1+b-a, \tag{64}$$

so from eqns (63) and (64) we obtain

$$f^*\omega = 2^{-(1/2n)}\left[\frac{A}{B}\left(a-b-\frac{1}{2n}\right)\right]^{a-b-(1/2n)}\left[\frac{t}{n(1-a+b)}\right]^{a-b}$$

$$\omega = 2^{-(1/n)}(1+b)\left[\frac{A}{B}\left(a-b-\frac{1}{2n}\right)\right]^{a-(1/n)}\left[\frac{t}{n(1-a+b)}\right]^{a}. \tag{65}$$

5. MATCHING OF SECTORS N AND E

Functions (R, θ) possess different forms in sectors N and E. Since φ, ψ and f, g represent the same quantities, they must be consistent to each other when $\theta \to \pi$ and $\xi \to \infty$. Now we consider the matching condition. Substituting eqn (35) into eqn (11) we obtain

$$R = r^{1-d} C_\varphi(\pi-\theta)$$
$$\Theta = r^c C_\psi(\pi-\theta)^{-1} \quad \text{when} \quad \theta \to \pi \quad \text{but} \quad \xi \to \infty, \tag{66}$$

and substituting eqn (61) into eqn (37) we obtain

$$R = r^{1-a+b} f^* \omega(\pi-\theta)$$
$$\Theta = r^a \frac{1}{\omega}(\pi-\theta)^{-1} \quad \text{when} \quad \theta \to \pi \quad \text{but} \quad \xi \to \infty. \tag{67}$$

Comparing eqns (66) and (67), we can see that eqns (11) and (37) are consistent with each other provided that

$$C_\varphi = f^* \omega, \quad C_\psi = \omega^{-1}. \tag{68}$$

On the other hand, noting eqn (45) and comparing eqn (36) with eqn (65), we find that eqn (68) is satisfied automatically. Therefore, the method of dividing the whole field into sectors N and E is successful.

6. THE RELATION BETWEEN LOAD AND FIELD

As mentioned above, the field, i.e. the magnitudes of φ, ψ, f and g are controlled by the parameters n, q, B/A and t. Now we derive the relation between the load and the parameters. Using eqn (26) and (28) we can calculate the resultant force $\mathscr{F}$ acting on the wedge tip,

$$\mathscr{F} = -2\int_0^\pi (\tau^{rr}\cos\theta - \tau^{r\theta}\sin\theta)\,\mathrm{d}\theta = tC \tag{69}$$

where

$$C = -\int_0^\pi \frac{\sin 2\theta}{(1-d)\varphi\varphi'}[(1-d)^2\varphi^2 - \varphi'^2]\,\mathrm{d}\theta. \tag{70}$$

Evidently, the value of coefficient C only depends on the shape of function φ but not its magnitude. On the other hand, from eqn (31) we can see the shape of function φ only depends on parameter n, q but not B/A. Therefore, we can conclude that C only depends on the values of n, q. The values of C are listed in Table 2 for various n and q.

7. CONCLUSIONS

For the rubber-like materials discussed in this paper, a notch corner under the compression of a line load will be locally closed. Under a similar load, the edge tip will be shrivelled to form a locally closed notch. In both cases, the deformation field contains a singular point (the wedge tip or the notch corner). When the singular point is approached,

Table 2. The values of C

n	q = 2	4	8
1	6.309	6.288	6.278
2	7.062	7.527	7.880
4	8.774	10.402	11.867
10	10.954	15.024	19.912

the stress possesses the order of r^{-1} (r is the distance to the singular point in the deformed configuration); this feature is similar to that of the linear elastic case.

The deformation pattern near a wedge tip (or a notch corner) can be described separately for expanding sector and narrowing sector. The mapping function $\varphi(\theta)$, $\psi(\theta)$ and $f(\xi)$, $g(\xi)$ are controlled by the constitutive parameters n, q, A, B and the loading parameter t that is dependent on the load $\mathscr{F}$. In particular the shape of function φ only depends on the parameters n and q.

The analysis in this paper is also valid for the problem that an infinite medium under the action of a concentrated line load, and the deformation pattern is the same as that for a wedge. It should be recalled that in linear elastic theory, the concentrated force problems for a wedge and for a whole plane have different forms of solution. However, in nonlinear theory, the solution forms are different for tension case and compression case.

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